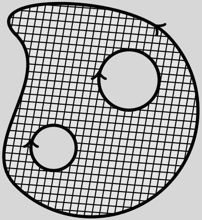
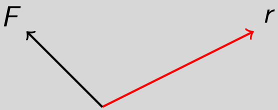
Green's Theorem
Chapters 5,6
Section 7.1

Daniel Barter



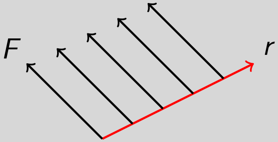
Work

Let $F = (F_1, F_2)$ be a force vector (newtons) and $r = (r_1, r_2)$ be a displacement vector (meters).

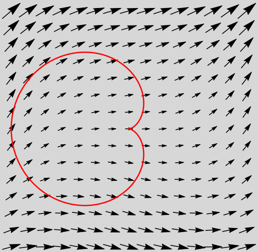


The work (newton meters = joules) done by the force F while a point particle is displaced by r is defined to be

$$F \cdot r = F_1 r_1 + F_2 r_2.$$



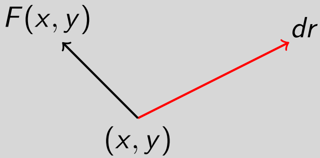
What if the particle doesn't move in a straight line and the force isn't constant?



$$F = (y^2 + 3, x^2 + y)$$

$$r = (x, y) = ((1 - \cos(t))\cos(t), (1 - \cos(t))\sin(t)) \quad 0 \le t \le 2\pi$$

Break the path up into small pieces



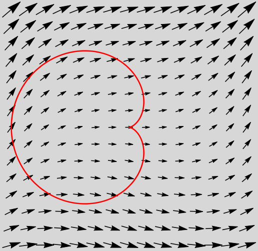
$$\begin{aligned} F(x, y) &= (F_1(x, y), F_2(x, y)) \\ dr &= (dx, dy) = (x'(t)dt, y'(t)dt) \\ F(x, y) \cdot dr &= F_1(x, y)dx + F_2(x, y)dy \end{aligned}$$

$$\text{work done} = F_1(x(t), y(t))x'(t)dt + F_2(x(t), y(t))y'(t)dt$$

Sum everything up

$$\text{total work done} = \int_a^b F_1(x(t), y(t))x'(t)dt + F_2(x(t), y(t))y'(t)dt$$

Example 1: How much work is done?



$$F = (y^2 + 3, x^2 + y)$$

$$r = ((1 - \cos(t))\cos(t), (1 - \cos(t))\sin(t)) \quad 0 \le t \le 2\pi$$

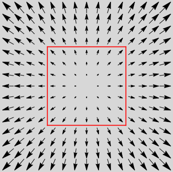
total work $= -5\pi/2 \approx -7.85398$

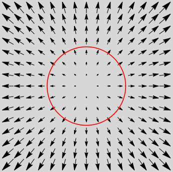
Example 2: How much work is done?

$$G = (x, y)$$

$$r1 = \begin{cases} (1, t) & -1 \leq t \leq 1 \\ (-t, 1) & -1 \leq t \leq 1 \\ (-1, -t) & -1 \leq t \leq 1 \\ (t, -1) & -1 \leq t \leq 1 \end{cases}$$

$$r2 = (\cos(t), \sin(t)) \quad 0 \leq t \leq 2\pi$$



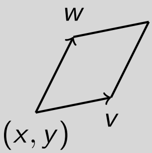


If $r(t)$ is a loop, then we have

$$\begin{aligned}
\int_r G \cdot dr &= \int_a^b x(t)x'(t) + y(t)y'(t)dt \\
&= \int_a^b r(t) \cdot r'(t)dt \\
&= \frac{1}{2}\int_a^b \frac{d}{dt}|r(t)|^2 dt \\
&= |r(b)|^2 - |r(a)|^2 = 0
\end{aligned}$$

Question: What makes the force field $G = (x, y)$ special compared to $F = (y^2 + 3, x^2 + y)$?

Green's Theorem: The fundamental idea

$P =$ 

$v = (v_1, v_2) \quad w = (w_1, w_2)$

$$W := \int_{\partial P} F_1(x, y)dx + F_2(x, y)dy = ??$$

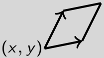
Green's Theorem: The fundamental idea

$$\begin{aligned} W \approx\ & F_1(x,y)v_1 + F_2(x,y)v_2 \\ & + F_1(x+v_1, y+v_2)w_1 \\ & + F_2(x+v_1, y+v_2)w_2 \\ & - F_1(x,y)w_1 - F_2(x,y)w_2 \\ & - F_1(x+w_1, y+w_2)v_1 \\ & - F_2(x+w_1, y+w_2)v_2 \end{aligned}$$

Green's Theorem: The fundamental idea

$$
\begin{aligned}
W \approx\ & F_1 v_1 + F_2 v_2 \\
& + \left(F_1 + \frac{\partial F_1}{\partial x} v_1 + \frac{\partial F_1}{\partial y} v_2 \right) w_1 \\
& + \left(F_2 + \frac{\partial F_2}{\partial x} v_1 + \frac{\partial F_2}{\partial y} v_2 \right) w_2 \\
& - F_1 w_1 - F_2 w_2 \\
& - \left(F_1 + \frac{\partial F_1}{\partial x} w_1 + \frac{\partial F_1}{\partial y} w_2 \right) v_1 \\
& - \left(F_2 + \frac{\partial F_2}{\partial x} w_1 + \frac{\partial F_2}{\partial y} w_2 \right) v_2 \\
& = \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) (v_1 w_2 - v_2 w_1)
\end{aligned}
$$

Green's Theorem: The fundamental idea

If $P =$ 

is very small, then

$$\int_{\partial P} F_1(x,y)dx + F_2(x,y)dy = \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)\mathrm{Area}(P)$$

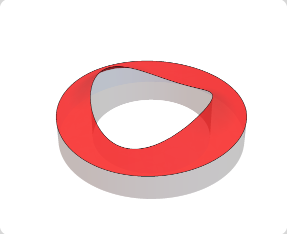
We define

$$\mathrm{curl}(F) = \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} = \det\begin{pmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \\ F_1 & F_2 \end{pmatrix}$$

Double Integrals

Suppose that $A \subseteq \mathbb{R}^2$ is a *closed* region and $f : A \to \mathbb{R}$ is a function. Then

$$\int_A f(x, y) dxdy = \text{Volume under the graph of } f.$$



$$f(x, y) = 4x^2 e^{-x^2-y^2} + 1 \qquad 2 \leq x^2 + y^2 \leq 5$$

We can take *closed* to mean that $\partial A \subseteq A$ in practice, but in theory, precisely defining *closed* is a subtle issue.

Changing Coordinates

$$\int_{2\leq x^2+y^2\leq 5}(4x^2e^{-x^2-y^2}+1)dxdy$$
$$=4\int_{2\leq x^2+y^2\leq 5}x^2e^{-x^2-y^2}dxdy+21\pi$$

We want to change to polar coordinates:

$$x=r\cos\theta \qquad y=r\sin\theta$$

$$dx=\cos\theta dr-r\sin\theta d\theta$$
$$dy=\sin\theta dr+r\cos\theta d\theta$$

Parallelogram rules

$$drdr = 0 \quad \text{(parallelogram has zero area)}$$
$$d\theta dr = -drd\theta \quad \text{(parallelogram has reverse orientation)}$$

$$\begin{aligned} dxdy &= (\cos\theta dr - r\sin\theta d\theta)(\sin\theta dr + r\cos\theta d\theta) \\ &= r\cos^2\theta drd\theta - r\sin^2\theta d\theta dr \\ &= r(\cos^2\theta + \sin^2\theta)drd\theta = rdrd\theta \end{aligned}$$

$$\int_{2\le x^2+y^2\le 5} x^2 e^{-x^2-y^2}\,dxdy = \int_0^{2\pi}\int_2^5 r^3 e^{-r^2}\cos^2\theta\,drd\theta$$

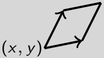
Green's Theorem

Let $A \subseteq \mathbb{R}^2$ be a closed region and F a vector field on A. Then

$$\int_{\partial A} F_1(x, y)dx + F_2(x, y)dy = \int_A \mathrm{curl}(F)dxdy$$

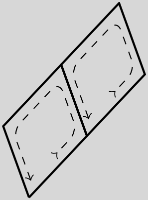
Important: You need to orient the boundary ∂A in the correct way! Boundary components for internal holes are oriented clockwise and outside boundary components are oriented counterclockwise.

Proof:

If $P =$ 

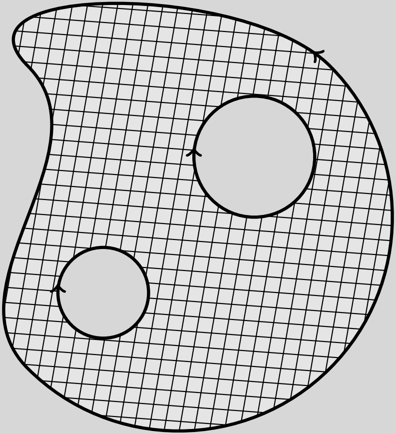
is very small, then

$$\int_{\partial P} F_1(x,y)dx + F_2(x,y)dy = \mathrm{curl}(F)\mathrm{Area}(P)$$



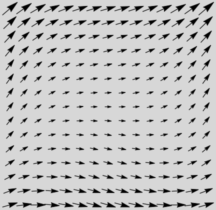
$$\int_{\gamma} F \cdot dr = -\int_{-\gamma} F \cdot dr$$

Proof:



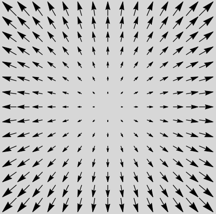
What makes G special compared to F

$F = (y^2 + 3, x^2 + y)$



$\mathrm{curl}(F) = 2x - 2y$

$G = (x, y)$



$\mathrm{curl}(G) = 0$

Example

Let $A \subseteq \mathbb{R}^2$ be a closed region. Then

$$\int_{\partial A} x dy = \int_A 1 dx dy = \text{area of } A$$

Therefore you can compute the area of A as a line integral around its boundary.

Potentials

Suppose that $A \subseteq \mathbb{R}^2$ is a closed region and $f : A \to \mathbb{R}$ is smooth function.

$\mathrm{curl}(\mathrm{grad}(f)) = 0$.

$(x, y) = \mathrm{grad}(x^2/2 + y^2/2)$.

Suppose that F is a vector field and $F = \mathrm{grad}(f)$. We call f a *potential* for F.

$$\begin{aligned} \mathrm{Work} &= \int_\gamma F \cdot dr = \int_\gamma \mathrm{grad}(f) \cdot dr = \int_a^b \mathrm{grad}(f)(\gamma(t)) \cdot \gamma'(t) dt \\ &= \int_b^a \frac{d}{dt} f(\gamma(t)) dt = f(\gamma(b)) - f(\gamma(a)) \end{aligned}$$

If a potential exists, work equals difference in potential.

Existence of potentials

Question: Suppose that F is a vector field on the closed region $A \subseteq \mathbb{R}^2$ and $\operatorname{curl}(F) = 0$. When does a potential exist?

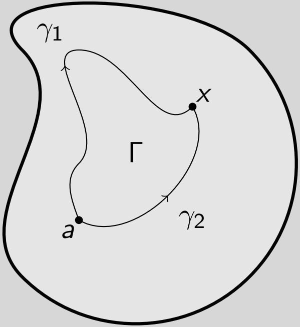
Potential formula: Fix $a \in A$. Then the potential is given by

$$f(x) = \int_\gamma F \cdot dr$$

where γ is a path in A from a to x.

Question: When is the right hand side independent of γ?

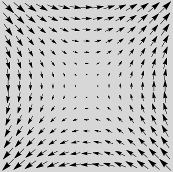
If A has no holes, then potentials always exist.



$$\int_{\gamma_2} F \cdot dr - \int_{\gamma_1} F \cdot dr = \int_{\Gamma} \mathrm{curl}(F)\,dxdy = 0$$

Example

Consider the vector field $F = (y, x)$.

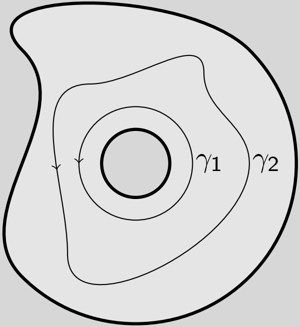


The potential is given by

$$f(a, b) = \int_{(0,0)}^{(a,b)} ydx + xdy$$

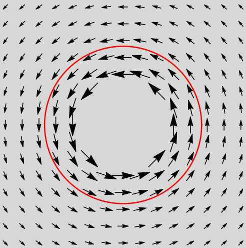
Using the path $x = ta, y = tb$ we get $f(a, b) = ab$.

If A has holes, then a potential *may not* exist.



$$\text{work around hole} = \int_{\gamma_1} F \cdot dr = \int_{\gamma_2} F \cdot dr$$

Example



$$F = \left(\frac{-y}{x^2 + y^2}, \frac{x}{x^2 + y^2} \right)$$

$$\gamma(t) = (\cos(t), \sin(t))$$

$\mathrm{curl}(F) = 0$. Potential doesn't exist because the force field does 2π work around the origin.

Theorem

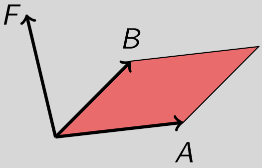
Suppose that F is a vector field on the closed region $A \subseteq \mathbb{R}^2$ and $\mathrm{curl}(F) = 0$. If the work done by F around each hole is zero, then a potential exists.

Higher Dimensional Generalizations of Green's Theorem
Chapters 7,8

Daniel Barter

Flux

Let $F = (F_1, F_2, F_3), A = (A_1, A_2, A_3)$ and $B = (B_1, B_2, B_3)$ be vectors.



The flux of F through the paralellagram $A \wedge B$ is exactly

$$F \cdot (A \times B) = \det(F, A, B) = \det \begin{pmatrix} F_1 & F_2 & F_3 \\ A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \end{pmatrix}$$

Recall: Green's Theorem

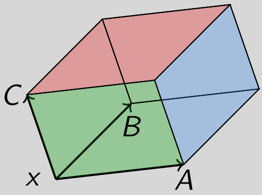
Let F be a vector field on $\mathbb{R}^2$ and $P =$  a very small paralellagram, then

$$\int_{\partial P} F_1(x,y)dx + F_2(x,y)dy = \mathrm{curl}(F)\mathrm{Area}(P).$$

Question: Is there an analog of Green's theorem for Flux?

Divergence Theorem: local version

Let F be a vector field on $\mathbb{R}^3$ and P be the parallelepiped spanned by the vectors A, B and C.



The flux of F through P (with an everywhere outward facing normal vector) is

$$\begin{aligned}\text{flux} = &\det(F(x), B, A) + \det(F(x + C), A, B)\\ &+ \det(F(x), A, C) + \det(F(x + B), C, A)\\ &+ \det(F(x), C, B) + \det(F(x + A), B, C)\end{aligned}$$

Divergence Theorem: local version

When P is very small, we have

$$\begin{aligned}\mathrm{flux} \approx & \det(DF(x)C, A, B)\\ & - \det(DF(x)B, A, C)\\ & + \det(DF(x)A, B, C)\\ & = \left(\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}\right)\mathrm{vol}(P)\end{aligned}$$

$$\mathrm{div}(F) = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}$$

So, the flux though the small parallelepiped P is exactly $\mathrm{div}(F)\mathrm{vol}(P)$.

Flux through a surface

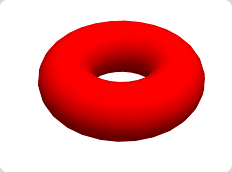
Let F be a vector field on $\mathbb{R}^3$ and $\Sigma \subseteq \mathbb{R}^3$ a surface. Choose a parameterization $x(s,t) = (x_1(s,t), x_2(s,t), x_3(s,t))$ of Σ. Then the flux of F through Σ is

$$\int_\Sigma F \cdot d\Sigma = \int_\Sigma \det\left(F(x(s,t)), \frac{\partial x}{\partial s}, \frac{\partial x}{\partial t}\right) dsdt$$

Example

Consider a torus $T \subseteq \mathbb{R}^3$. We can parameterize it by

$$x_1(\theta, \phi) = (2 + \cos\theta)\cos\phi$$
$$x_2(\theta, \phi) = (2 + \cos\theta)\sin\phi$$
$$x_3(\theta, \phi) = \sin\theta$$



The flux of $F = (F_1, F_2, F_3)$ through T is

$$\int_{-\pi}^{\pi}\int_0^{2\pi} \det\left(F, \frac{\partial x}{\partial\theta}, \frac{\partial x}{\partial\phi}\right) d\theta d\phi = 0$$

Divergence Theorem: global version

Let F be a vector field on $\mathbb{R}^3$ and $A \subseteq \mathbb{R}^3$ a closed 3-dimensional region with boundary the surface Σ. Then

$$\int_{\Sigma} F \cdot d\Sigma = \int_{A} \operatorname{div}(F)\, dxdydz$$

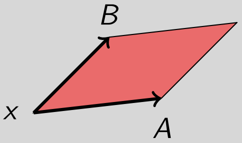
Example

Let $F = (x/3, y/3, z/3)$, $A \subseteq \mathbb{R}^3$ be a closed 3-dimensional region and $\Sigma = \partial A$. Then $\mathrm{div}(F) = 1$ so

$$\text{Volume of } A = \int_A 1 dxdydz = \int_\Sigma F \cdot d\Sigma$$

3D version of Green's Theorem: local version

Let F be a vector field on $\mathbb{R}^3$ and choose a small parallelogram $P = A \wedge B$.



The work done by F around P is

$$\begin{aligned}
&F(x) \cdot A + F(x+A) \cdot B - F(x) \cdot B - F(x+B) \cdot A \\
&= (DF(x)A) \cdot B - (DF(x)B) \cdot A \\
&= \det(\operatorname{curl}(F), A, B)
\end{aligned}$$

The work done by F around P is the flux of $\operatorname{curl}(F)$ through P.

3D version of Green's Theorem: global version

Let F be a vector field on $\mathbb{R}^3$ and $\Sigma \subseteq \mathbb{R}^3$ a surface with boundary curve γ. Then

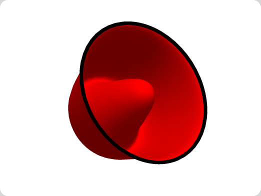
$$\int_\gamma F \cdot d\gamma = \int_\Sigma \mathrm{curl}(F) \cdot d\Sigma$$

where

$$\mathrm{curl}(F) = \det \begin{pmatrix} e_1 & e_2 & e_3 \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ F_1 & F_2 & F_3 \end{pmatrix}$$

Example

Consider the surface $(\sin(\theta)\cos(\phi), \sin(\theta)\sin(\phi), 2\cos^2(\theta)\sin^3(2\theta))$ bounded by $(\cos(t), \sin(t), 0)$.



Compute the work done by $F = (-y, x, 0)$ around the boundary circle.

Moral of the Story

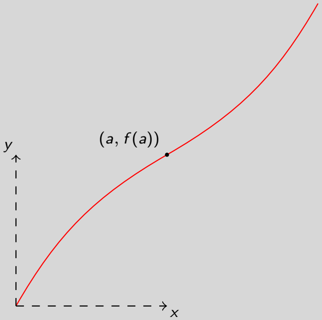
Whenever you have a field that can be integrated over d-dimensional parallelograms, you should integrate it over the boundary of a small $d + 1$-dimensional parallelogram. This way, you can discover the various different versions of Green's theorem as you need them without having to memorize lots of complicated formulas.

There is a common generalization of all these theorem's called Stoke's Theorem. Come to office hours if you want to learn about it :D

Taylor Polynomials and Series

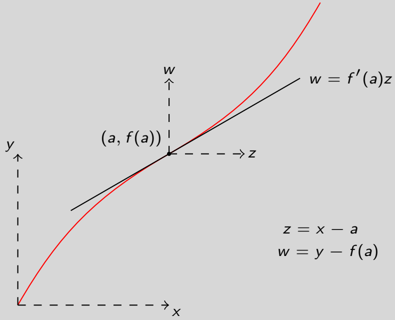
The Derivative as a Linear Approximation

Consider a function $f : \mathbb{R} \to \mathbb{R}$. The graph $y = f(x)$ looks something like



The Derivative as a Linear Approximation

The derivative $f'(a)$ gives us the best linear approximation to f at $(a, f(a))$:



Notice that $w = f'(a)z$ is just the equation $y - f(a) = f'(a)(x - a)$ using the coordinate system z, w which is centered at the point $(a, f(a))$.

Degree d Polynomial Approximation

What is the best degree d polynomial approximation to f at $(a, f(a))$?

$$
\begin{aligned}
w = T_{a,f,d}(z) &= \sum_{k=1}^{d} \frac{f^{(k)}(a)}{k!} z^k \\
&= f'(a)z + \frac{f''(a)}{2!} z^2 + \cdots + \frac{f^{(d)}(a)}{d!} z^d
\end{aligned}
$$

$$
y = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(d)}(a)}{d!}(x-a)^d
$$

We call $T_{a,f,d}(z)$ the degree d Taylor polynomial for f at a.

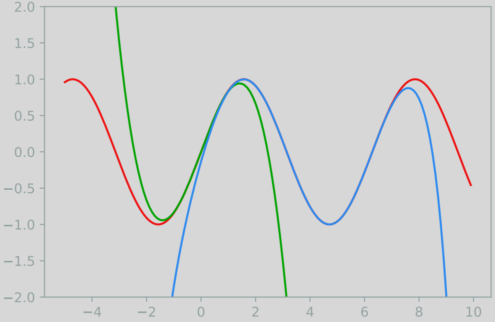
Example

Taylor polynomial for $\sin(x)$:

degree 3 at $(0,0) \rightsquigarrow x - x^3/6$

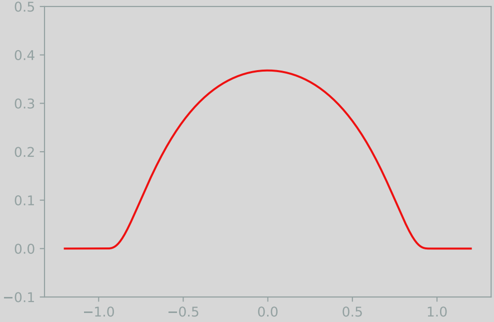
degree 9 at $(4, -0.756802) \rightsquigarrow$

$-0.132216 + 1.30689x - 0.312849x^2 + 0.0151476x^3 - 0.0647837x^4 + 0.0220721x^5 - 0.00130555x^6 - 0.000307201x^7 + 0.0000460757x^8 - 1.80127 \cdot 10^{-6}x^9$



Example: Bump Function

$$b(x) = \begin{cases} \exp\left(-\frac{1}{1-x^2}\right) & x \in (-1,1) \\ 0 & \text{otherwise} \end{cases}$$



The taylor polynomial at 1 is 0 for all degrees. Taylor polynomials don't always behave the way you would expect...

The Chain Rule

Recall the chain rule:

$$(g \circ f)'(a) = g'(f(a))f'(a)$$

We can rephrase this as

$$T_{a,g\circ f,1}(z) = T_{f(a),g,1}(T_{a,f,1}(z))$$

where

$$T_{a,f,d}(z) = \sum_{k=1}^{d} \frac{f^{(k)}(a)}{k!} z^k$$

The Chain Rule for Higher Derivatives?!

$$T_{a,f,d}(z) = \sum_{k=1}^{d} \frac{f^{(k)}(a)}{k!} z^k$$

$$T_{a,g\circ f,d}(z) = T_{f(a),g,d}(T_{a,f,d}(z)) + O(z^{d+1})$$

The $O(z^{d+1})$ means that we just forget the terms which have order z^{d+1} or higher.

The Chain Rule for Higher Derivatives?!

$$
\begin{aligned}
T_{a,g\circ f,1}(z) &= z\,\mathrm{f}^{(1)}(a)\,\mathrm{g}^{(1)}(f(a)) \\
T_{a,g\circ f,2}(z) &= z\,\mathrm{f}^{(1)}(a)\,\mathrm{g}^{(1)}(f(a)) \\
&\quad + z^2\left(\frac{\mathrm{f}^{(1)^2}(a)\,\mathrm{g}^{(2)}(f(a))}{2} + \frac{\mathrm{f}^{(2)}(a)\,\mathrm{g}^{(1)}(f(a))}{2}\right) \\
T_{a,g\circ f,3}(z) &= z\,\mathrm{f}^{(1)}(a)\,\mathrm{g}^{(1)}(f(a)) \\
&\quad + z^2\left(\frac{\mathrm{f}^{(1)^2}(a)\,\mathrm{g}^{(2)}(f(a))}{2} + \frac{\mathrm{f}^{(2)}(a)\,\mathrm{g}^{(1)}(f(a))}{2}\right) \\
&\quad + z^3\left(\frac{\mathrm{f}^{(1)^3}(a)\,\mathrm{g}^{(3)}(f(a))}{6} + \frac{\mathrm{f}^{(1)}(a)\,\mathrm{f}^{(2)}(a)\,\mathrm{g}^{(2)}(f(a))}{2} + \frac{\mathrm{f}^{(3)}(a)\,\mathrm{g}^{(1)}(f(a))}{6}\right) \\
&\quad \vdots
\end{aligned}
$$

The Chain Rule for Higher Derivatives?!

$$
\begin{aligned}
(g \circ f)^{(1)}(a) &= f^{(1)}(a)g^{(1)}(f(a)) \\
(g \circ f)^{(2)}(a) &= f^{(1)}(a)f^{(1)}(a)g^{(2)}(f(a)) \\
&\quad + f^{(2)}(a)g^{(1)}(f(a)) \\
(g \circ f)^{(3)}(a) &= f^{(1)}(a)f^{(1)}(a)f^{(1)}(a)g^{(3)}(f(a)) \\
&\quad + 3f^{(1)}(a)f^{(2)}(a)g^{(2)}(f(a)) \\
&\quad + f^{(3)}(a)g^{(1)}(f(a)) \\
&\vdots
\end{aligned}
$$

Taylor Expansions for Solutions to Ordinary Differential Equations

Consider the differential equation

$$f'(x) = g(f(x)) \quad f(x_0) = y_0$$

Differentiating i times gives us

$$f^{(i+1)}(a) = (g \circ f)^{(i)}(a)$$

Using the chain rule for higher derivatives gives us:

Taylor Expansions for Solutions to Ordinary Differential Equations: Butcher Series

$$
\begin{aligned}
f^{(1)}(a) &= g(f(a)) \\
f^{(2)}(a) &= g(f(a))\,\mathrm{g}^{(1)}(f(a)) \\
f^{(3)}(a) &= g^2(f(a))\,\mathrm{g}^{(2)}(f(a)) + g(f(a))\,\mathrm{g}^{(1)^2}(f(a)) \\
f^{(4)}(a) &= g^3(f(a))\,\mathrm{g}^{(3)}(f(a)) + 4g^2(f(a))\,\mathrm{g}^{(1)}(f(a))\,\mathrm{g}^{(2)}(f(a)) + g(f(a))\,\mathrm{g}^{(1)^3}(f(a)) \\
f^{(5)}(a) &= g^4(f(a))\,\mathrm{g}^{(4)}(f(a)) + 7g^3(f(a))\,\mathrm{g}^{(1)}(f(a))\,\mathrm{g}^{(3)}(f(a)) + 4g^3(f(a))\,\mathrm{g}^{(2)^2}(f(a)) \\
&\quad + 11g^2(f(a))\,\mathrm{g}^{(1)^2}(f(a))\,\mathrm{g}^{(2)}(f(a)) + g(f(a))\,\mathrm{g}^{(1)^4}(f(a)) \\
f^{(6)}(a) &= g^5(f(a))\,\mathrm{g}^{(5)}(f(a)) + 11g^4(f(a))\,\mathrm{g}^{(1)}(f(a))\,\mathrm{g}^{(4)}(f(a)) + 15g^4(f(a))\,\mathrm{g}^{(2)}(f(a))\,\mathrm{g}^{(3)}(f(a)) \\
&\quad + 32g^3(f(a))\,\mathrm{g}^{(1)^2}(f(a))\,\mathrm{g}^{(3)}(f(a)) + 34g^3(f(a))\,\mathrm{g}^{(1)}(f(a))\,\mathrm{g}^{(2)^2}(f(a)) \\
&\quad + 26g^2(f(a))\,\mathrm{g}^{(1)^3}(f(a))\,\mathrm{g}^{(2)}(f(a)) + g(f(a))\,\mathrm{g}^{(1)^5}(f(a)) \\
&\ \ \vdots
\end{aligned}
$$

Taylor Expansions for Solutions to Ordinary Differential Equations

The general solution to $f'(x) = g(f(x))$ is given by

$$f(a+x) = \sum_{i=0}^{\infty} \frac{f^{(i)}(a)x^i}{i!}.$$

Question: When does this infinite sum make sense?

The general theory of sequences and series is a useful tool for answering this kind of question.

Sequences

A sequence is a never ending list of numbers

$$a_0, a_1, a_2, \cdots$$

$1, 2, 3, 4, \cdots$

$1, -1, 1, -1, \cdots$

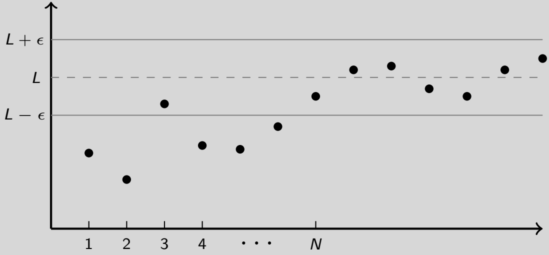
$1, 0.1, 0.01, 0.001, \cdots$

$1, 1/2, 1/3, 1/4, \cdots$

The limit of a sequence

Informally, we say that the sequence a_n converges to L if a_n gets closer and closer to L as n gets larger and larger.

Formally, we say that $\lim_{n\to\infty} a_n = L$ if for all $\epsilon > 0$, there exists an N such that for all $n > N$, we have $|a_n - L| < \epsilon$.



Properties of limits

Suppose that both limits $\lim_{n\to\infty} a_n$ and $\lim_{n\to\infty} b_n$ exist and are finite. Then

$$
\begin{aligned}
\lim_{n\to\infty}(a_n + b_n) &= \lim_{n\to\infty} a_n + \lim_{n\to\infty} b_n \\
\lim_{n\to\infty}(a_n - b_n) &= \lim_{n\to\infty} a_n - \lim_{n\to\infty} b_n \\
\lim_{n\to\infty}(a_n b_n) &= \lim_{n\to\infty} a_n \lim_{n\to\infty} b_n \\
\lim_{n\to\infty}(a_n / b_n) &= \lim_{n\to\infty} a_n / \lim_{n\to\infty} b_n, \lim_{n\to\infty} b_n \neq 0
\end{aligned}
$$

Examples

$$\frac{n}{n+1} = \frac{1}{1+(1/n)} \to 1 \ \text{ as } n \to \infty$$

$$\frac{n}{2n+3} = \frac{1}{2+(3/n)} \to \frac{1}{2} \ \text{ as } n \to \infty$$

Monotone Convergence Theorem

Suppose that we have a sequence a_i which is increasing $a_0 \leq a_1 \leq a_2 \leq \cdots$ and bounded: there exists some B such that $a_i < B$ for all i. Then the limit $\lim_{n\to\infty} a_n$ exists.

Comes for free when constructing $\mathbb{R}$ from $\mathbb{Q}$.

$$1, 1.4, 1.41, 1.414, 1.4142, 1.41421, 1.414213, \cdots \to \sqrt{2}$$

Orders of Growth

Given two functions $f(n) > 0$ and $g(n) > 0$ we write

$f(n) \ll g(n)$ if $lim_{n \to \infty} f(n)/g(n) = 0$.

$f(n) \equiv g(n)$ if $lim_{n \to \infty} f(n)/g(n) = c$ for some $0 < c < \infty$

$f(n) \gg g(n)$ if $lim_{n \to \infty} f(n)/g(n) = \infty$

For example, if we have two degree p polynomials

$$f(n) = a_p n^p + \cdots + a_1 n + a_0$$
$$g(n) = b_p n^p + \cdots + b_1 n + b_0$$

then $f(n) \equiv g(n)$. We call a set of functions equivalent under $\equiv$ an *order of growth*. We tend to represent an order of growth by the simplest function in the class, for example n^p in the above example.

Orders of Growth

$$
\begin{gathered}
0 \ll 1 \ll \log(\log(n)) \ll \log(n) \ll \\
n \ll n\log(n) \ll n^2 \ll \cdots \ll n^p \ll \\
\cdots \ll e^n \ll n! \ll n^n \ll n^{n^n} \ll \cdots
\end{gathered}
$$

Series

A series is an infinite sum

$$\sum_{i=0}^{\infty} a_i$$

Associated to a series is the sequence of partial sums

$$\begin{aligned} S_0 &= a_0 \\ S_1 &= a_0 + a_1 \\ S_2 &= a_0 + a_1 + a_2 \\ &\vdots \\ S_n &= \sum_{i=0}^{n} a_i \end{aligned}$$

Series

From the sequence of partial sums,

$$S_n = \sum_{i=0}^{n} a_i$$

we define the value of the series as the limit of the sequence of partial sums:

$$\sum_{i=0}^{\infty} a_i = \lim_{n \to \infty} S_n$$

The Most Important Series

The most important series is the geometric series:

$$\sum_{i=0}^{\infty} r^i$$

Consider the partial sum

$$S_n = \sum_{i=0}^{n} r^i$$

We have $S_n - rS_n = 1 - r^{n+1}$ which implies

$$S_n = \frac{1 - r^{n+1}}{1 - r}$$

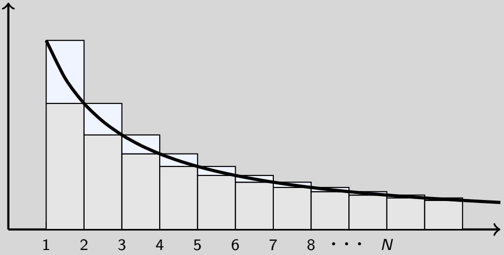
The Most Important Series

Taking the limit as $n \to \infty$ gives

$$\sum_{i=0}^{\infty} r^i = \frac{1}{1-r} \text{ when } |r| < 1$$

Series and Integration

Suppose that we have a decreasing function $f(x) > 0$:



$$\sum_{n=2}^{\infty} f(n) < \int_1^{\infty} f(x)dx < \sum_{n=1}^{\infty} f(n)$$

$\sum_n^{\infty} f(n)$ converges if and only if $\int_1^{\infty} f(x)dx$ converges.

Examples

$\int_1^\infty \frac{dx}{x^2} = 1$ so $\sum_{n=1}^\infty \frac{1}{n^2}$ converges. Infact, $\sum_{n=1}^\infty \frac{1}{n^2} = \pi^2/6$ as we will see when we get to fourier series.

$\int_1^\infty \frac{dx}{x} = \infty$ so $\sum_{n=1}^\infty \frac{1}{n}$ diverges.

$\int_1^\infty \frac{dx}{2x(2x+1)} = \ln(3/2)/2$ so $\sum_{n=1}^\infty \frac{1}{2n(2n+1)}$ converges. Infact, the value is $1 - \ln(2)$ (as can be checked using a computer), but I don't know a good explination for this.

The Divergence test

If we have a series $\sum_n a_n$ which converges to a limit L, then $\lim_{n\to\infty} a_n = 0$.

$$a_n = S_n - S_{n-1} \to 0$$

where $S_n = a_0 + \cdots + a_n$.

The Comparison Test

Suppose we have sequences $a_n \geq 0$ and $b_n \geq 0$ and $\lim_{n\to\infty} a_n/b_n = c < \infty$. Then $\sum_n a_n$ converges if and only if $\sum_n b_n$ converges.

idea: Once n is large enough, we have $cb_n/2 \leq a_n \leq 2cb_n$.

We can use the comparison test to check the convergence of series like

$$\sum_n \frac{2n^3 + 4n + 1}{5n^5 + 7n^4 + 2n + 1}.$$

$$\frac{2n^3 + 4n + 1}{5n^5 + 7n^4 + 2n + 1} / \frac{1}{n^2} \to 2/5$$

Absolute Convergence

We say that a series $\sum_i^\infty a_i$ converges absolutely if $\sum_i^\infty |a_i|$ converges.

Consider the series $\sum_{n=1}^\infty (-1)^{n+1}/n$. It doesn't converge absolutely, but

$$\sum_{n=1}^\infty \frac{(-1)^{n+1}}{n} = 1 - \sum_{n=1}^\infty \frac{1}{2n(2n+1)} = \ln(2)$$

so it does converge. We call this **conditional convergence**

Absolute convergence is stronger than condition convergence.

What does absolute convergence mean?

If a series $\sum a_i$ converges absolutely, then every permutation $\sum a_{\sigma(i)}$ also converges to the same value.
If a series $\sum a_i$ converges conditionally, we can rearange the terms to converge to any value:

$$\sum_n \frac{(-1)^{n+1}}{n} = \sum_n \frac{1}{2n+1} - \sum_n \frac{1}{2n}$$

Both series appearing on the right hand side diverge.

The ratio test

If $\sum a_n$ is a series and $\lim_{n\to\infty}|a_{n+1}/a_n| < 1$, then $\sum a_n$ converges absolutely.

Idea: Eventually, $|a_{n+1}/a_n| < r < 1$ and $\sum_n r^n$ converges when $r < 1$.

The series $\sum_n 1/n!$ converges because

$$\frac{1/(n+1)!}{1/n!} = \frac{1}{n+1} \to 0$$

Infact, $\sum_n 1/n! = e$ as we shall soon see.

Power Series

We call the series $\sum_n a_n z^n$ a **power series**. By the ratio test, it converges absolutely when

$$\lim_{n\to\infty}\left|\frac{a_{n+1}z^{n+1}}{a_n z^n}\right| = |z| \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| < 1.$$

If we set

$$R = \frac{1}{\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|},$$

the power series $\sum_n a_n z^n$ converges absolutely when $|z| < R$.

We call R the **radius of convergence**.

caution! For some pathalogical examples, the limit in R doesn't exist. There is a different test called the **root test** that gives a formula for R which always exists.

Central Example

The power series

$$f(z) = \sum_{n=0}^{\infty} \frac{z^n}{n!}$$

has radius of convergence $R = \infty$.

We can check that $f'(z) = f(z)$ and $f(0) = 1$. Therefore $f(z) = e^z$.

Trigonometric Functions

$$\sin(t) = \sum_{n=0}^{\infty} \frac{(-1)^n t^{2n+1}}{(2n+1)!} \quad t \in \mathbb{R}$$

$$\cos(t) = \sum_{n=0}^{\infty} \frac{(-1)^n t^{2n}}{(2n)!} \qquad t \in \mathbb{R}$$

More Examples

$$\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n \qquad |z| < 1$$

$$\frac{1}{1+z^2} = \sum_{n=0}^{\infty} (-1)^n z^{2n} \qquad |z| < 1$$

These are taylor expansions at the origin. It is clear why the first has radius of convergence 1, but not so clear for the second... This will become clear after we start talking about complex numbers.

The Complex Plane

History

Consider the quadratic equation $f(z) = z^2 + bz + c = 0$. It has been known for more than 4000 years that

$$z = \frac{-b \pm \sqrt{b^2 - 4c}}{2}.$$

There are three cases:

$b^2 - 4c > 0$: The equation has two distinct roots.

$b^2 - 4c = 0$: Then $f(z) = (z + b/2)^2$, so we should count the root $-b/2$ twice.

$b^2 - 4c < 0$: There are no solutions.

The simplest example where case 3 occours is $f(z) = z^2 + 1$.

$i = \sqrt{-1}$

We can imagine taking the real numbers $\mathbb{R}$ and formally adjoining $i = \sqrt{-1}$. The resulting number system is called the complex numbers and denoted by $\mathbb{C}$.

A general complex number is of the form

$$a + ib \qquad a, b \in \mathbb{R}.$$

$$\begin{aligned} (a_1 + ib_1) + (a_2 + ib_2) &= (a_1 + a_2) + i(b_1 + b_2) \\ (a_1 + ib_1)(a_2 + ib_2) &= (a_1 a_2 - b_1 b_2) + i(a_1 b_2 + a_2 b_1) \\ 1/(a + ib) &= \frac{a}{a^2 + b^2} - i\frac{b}{a^2 + b^2} \end{aligned}$$

Fundamental Theorem of Algebra

Let $f(z)$ be a degree d polynomial. In the 1800s it was proved that f has exactly d complex roots, when you count them with the correct multiplicity.

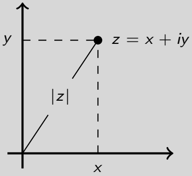
In otherwords, we can write

$$f(z) = (z - r_1) \cdots (z - r_d) \qquad r_i \in \mathbb{C}$$

Once you add $\sqrt{-1}$ to your number system, you can fully solve every polynomial equation.

The Complex Plane

A complex number looks like $z = x + iy$ where $x, y \in \mathbb{R}$. Therefore, we should visualize complex numbers as living in the plane:



$$|z| = \sqrt{x^2 + y^2}$$

Complex Conjugation

An important operation on complex numbers is complex conjugation:

$$\overline{z} = x - iy.$$

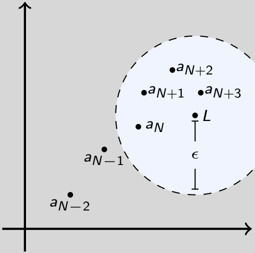
Complex conjugation swaps the two solutions of $z^2 + 1$.

Geometrically, it corresponds to reflection in the x-axis: $(x, y) \mapsto (x, -y)$. Also, notice that

$$z\overline{z} = |z|^2.$$

Sequences of Complex Numbers

We can have a sequence of complex numbers $a_0, a_1, a_2, \cdots$



For all $\epsilon > 0$, there exists an $N > 0$ such that $|a_n - L| < \epsilon$ for all $n \geq N$.

Sequences of Complex Numbers

Everything we learned about sequences and series of real numbers carries over to complex numbers.

If $z_n = x_n + iy_n$, then

$$\lim_{n\to\infty} z_n = \lim_{n\to\infty} x_n + i \lim_{n\to\infty} y_n$$

A power series

$$f(z) = \sum_{n=0}^{\infty} a_n z^n$$

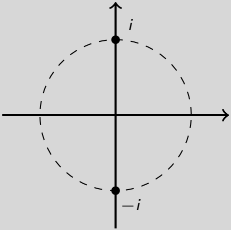
converges for all $z \in \mathbb{C}$ with $|z| <$ radius of convergence.

Solving a mystery

Recall that

$$\frac{1}{1+z^2} = \sum_{n=0}^{\infty} (-1)^n z^{2n} \quad |z| < 1$$

On the complex plane, the solutions of $z^2 + 1$ look like



The exponential function

We can define the complex exponential function

$$e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!} \quad z \in \mathbb{C}$$

The radius of convergence for this series was ∞, so the series converges absolutely for all complex numbers.

If z is real, this agrees with the exponential function you are all familiar with.

We have $e^{z+w} = e^z e^w$ for all $z, w \in \mathbb{C}$.

The exponential function

$$
\begin{aligned}
e^{it} &= \sum_{n=0}^{\infty} \frac{(it)^n}{n!} = \sum_{n=0}^{\infty} \frac{(it)^{2n}}{(2n)!} + \sum_{n=0}^{\infty} \frac{(it)^{2n+1}}{(2n+1)!} \\
&= \sum_{n=0}^{\infty} \frac{(-1)^n t^{2n}}{(2n)!} + i \sum_{n=0}^{\infty} \frac{(-1)^n t^{2n+1}}{(2n+1)!} \\
&= \cos(t) + i \sin(t)
\end{aligned}
$$

Notice the formula

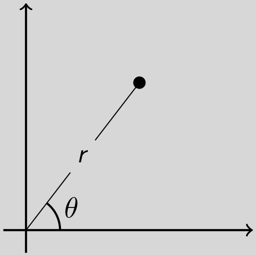
$$e^{i\pi} = -1$$

which is somewhat famous.

The exponential function

$$e^{x+iy} = e^x e^{iy}$$

We can use the exponential function to express every complex number in the following form: $re^{i\theta}$ for $r > 0$.



Notice that

$$re^{i\theta} = re^{i(\theta+2\pi)}.$$

Geometric intuition for complex multiplication

$$(r_1 e^{i\theta_1})(r_2 e^{i\theta_2}) = r_1 r_2 e^{i(\theta_1+\theta_2)}$$

For example, multiplication by i is rotation by $\pi/2$.

$$\frac{1}{re^{i\theta}} = \frac{e^{-i\theta}}{r}$$

Fourier Series

Periodic Functions

A function $f : \mathbb{R} \to \mathbb{C}$ is periodic with period ω if $f(t + n\omega) = f(t)$ for all $n \in \mathbb{Z}$.

The prototypical example is

$$e_k(t) = \exp\left(\frac{2\pi k}{\omega} ti\right) = \cos\left(\frac{2\pi k}{\omega} t\right) + i \sin\left(\frac{2\pi k}{\omega} t\right)$$

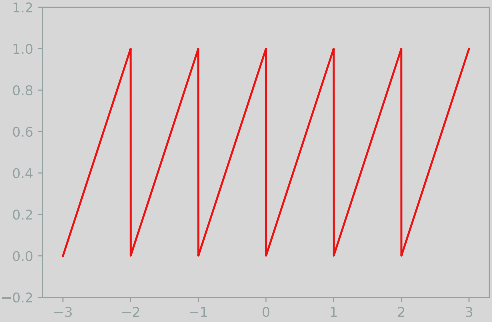
The function e_k is very special:

$$e_k(s + t) = e_k(s)e_k(t)$$

This is not true of sin and cos independently, which is the main reason we want to bundle sin and cos up into a single complex valued function.

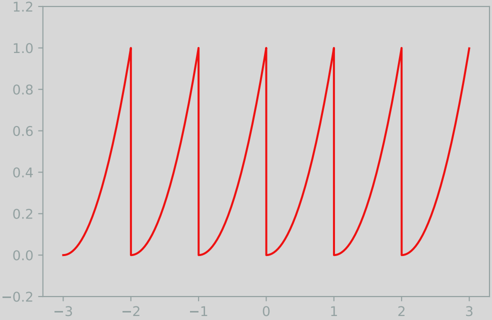
There are lots of interesting periodic functions!

The linear saw function: $f(x) = x \quad 0 \leq x \leq 1$



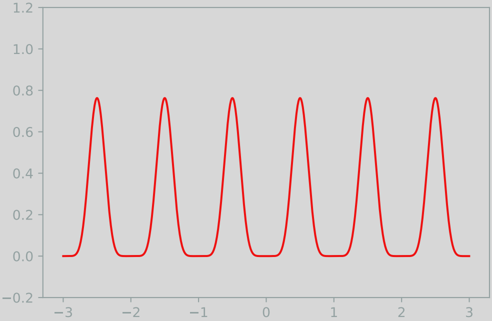
There are lots of interesting periodic functions!

The quadratic saw function: $f(x) = x^2 \quad 0 \leq x \leq 1$



There are lots of interesting periodic functions!

this thing: $f(x) = 200000x^9(x-1)^9 \quad 0 \leq x \leq 1$



There are lots of interesting periodic functions!

The Peter–Weyl Theorem

If $f : \mathbb{R} \to \mathbb{C}$ is a periodic function with period ω, then we can "represent" f as an infinite sum of our prototypical periodic functions:

$$f(t) = \sum_{k=-\infty}^{\infty} a_k \exp\left(\frac{2\pi k}{\omega} ti\right) = \sum_{k=-\infty}^{\infty} a_k e_k(t)$$

Understanding what is meant by "represent" and when/how this series converges is extremely subtle. This was the reason that humanity decided to develop rigorous foundations for calculus in the 1800s.

How to compute the a_ks

$$\frac{1}{\omega}\int_0^{\omega} e_k(t)\overline{e_m(t)}dt = \begin{cases} 0 & m \neq k \\ 1 & m = k \end{cases}$$

$$\frac{1}{\omega}\int_0^{\omega} f(t)\overline{e_m(t)}dt = \sum_{k=-\infty}^{\infty} a_k \frac{1}{\omega}\int_0^{\omega} e_k(t)\overline{e_m(t)}dt = a_m$$

So we have

$$a_m = \frac{1}{\omega}\int_0^{\omega} f(t)\overline{e_m(t)}dt$$

What happens when f is real valued?

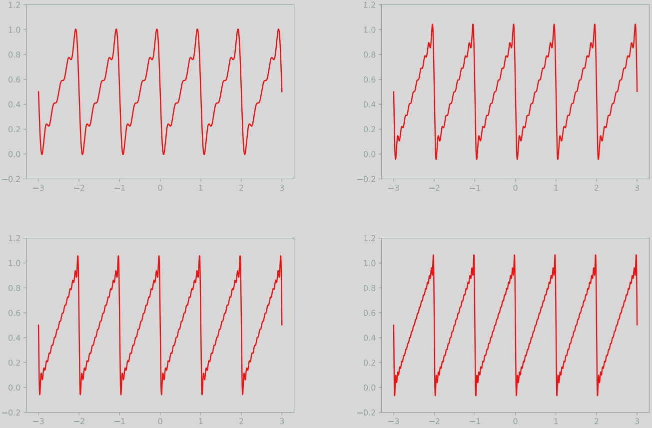
$$\overline{a_m} = \frac{1}{\omega}\int_0^\omega f(t)e_m(t)dt = \frac{1}{\omega}\int_0^\omega f(t)\overline{e_{-m}(t)}dt = a_{-m}$$

If we allow ourselves to rearange the fourier series, then

$$f(t) = a_0 + \sum_{k=1}^{\infty} a_k e_k(t) + \overline{a_k e_k(t)}$$

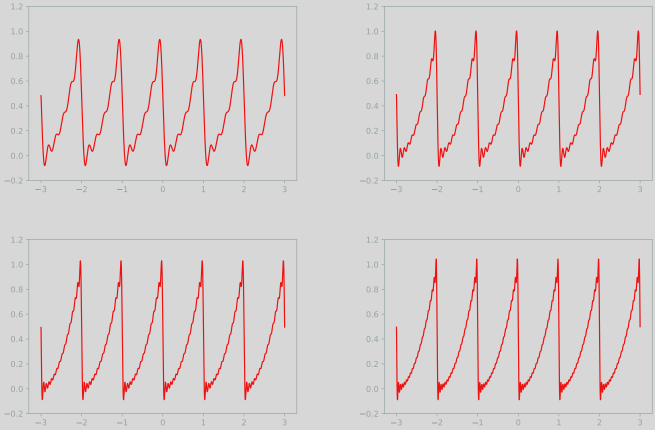
which is real valued.

Linear saw function: 5, 10, 15 and 20 terms

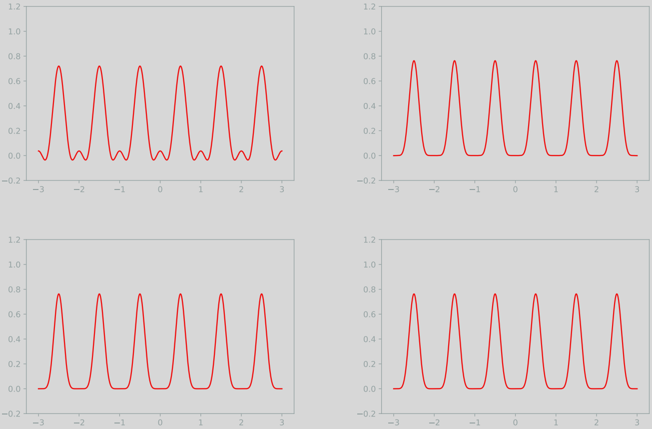


The wiggly parts at the ends are called Gibbs oscillations.

Quadratic saw function: 5, 10, 15 and 20 terms



Smoothed saw function: 2, 4, 6 and 8 terms



Much more Interesting Examples

https://www.youtube.com/watch?v=-qgreAUpPwM

http://www.theory.physics.ubc.ca/341-current/pluck/pluck.html

The Derivative of a Periodic Function

If we have a periodic function

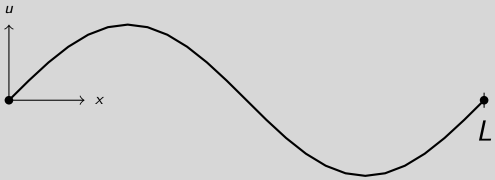
$$f(x) = \sum_{k=-\infty}^{\infty} a_k \exp\left(\frac{2\pi k}{\omega} xi\right)$$

then we can compute the derivative as follows:

$$f'(x) = \sum_{k=-\infty}^{\infty} \frac{2\pi ki}{\omega} a_k \exp\left(\frac{2\pi k}{\omega} xi\right)$$

The Wave Equation

We want to model the vibration of a guitar string:



The system is governed by the differential equation

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$$

A nice derivation is given in Dave Brenson's book on the mathematics of music: `https://homepages.abdn.ac.uk/d.j.benson/pages/html/music.pdf`

Solution using Fourier Series

First take a Fourier expansion in the x variable:

$$u(x,t) = \sum_{k=-\infty}^{\infty} a_k(t) \exp\left(\frac{2\pi k}{L} ix\right)$$

Then the wave equation becomes

$$\sum_{k=-\infty}^{\infty} a_k''(t) \exp\left(\frac{2\pi k}{L} ix\right) = \sum_{k=-\infty}^{\infty} a_k(t) c^2 \left(\frac{2\pi ki}{L}\right)^2 \exp\left(\frac{2\pi k}{L} ix\right)$$

For each k we have

$$a_k''(t) = -\left(\frac{2\pi kc}{L}\right)^2 a_k(t)$$

Solution using Fourier Series

Solving this differential equation gives

$$a_k(t) = A_k \exp\left(\frac{2\pi kc}{L} it\right) + B_k \exp\left(-\frac{2\pi kc}{L} it\right)$$

The general solution looks like

$$u(x,t) =$$
$$\sum_{k=-\infty}^{\infty} \left[A_k \exp\left(\frac{2\pi kc}{L} it\right) + B_k \exp\left(-\frac{2\pi kc}{L} it\right)\right] \exp\left(\frac{2\pi k}{L} ix\right)$$

Notice that the initial condition is

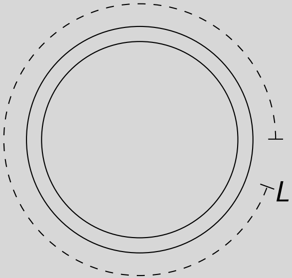
$$u(x,0) = \sum_{k=-\infty}^{\infty} (A_k + B_k) \exp\left(\frac{2\pi k}{L} ix\right)$$

The Heat Equation

Another equation which shows up regularly in physics is the heat equation:

$$\frac{\partial u}{\partial t} = \alpha \frac{\partial^2 u}{\partial x^2}$$

It controls the diffusion of heat through a metal loop:



First take a Fourier expansion in the x variable:

$$u(x,t) = \sum_{k=-\infty}^{\infty} a_k(t) \exp\left(\frac{2\pi k}{L} ix\right)$$

The heat equation becomes

$$\sum_{k=-\infty}^{\infty} a_k'(t) \exp\left(\frac{2\pi k}{L} ix\right) = \sum_{k=-\infty}^{\infty} -\alpha \frac{4\pi^2 k^2}{L^2} a_k(t) \exp\left(\frac{2\pi k}{L} ix\right)$$

This gives

$$a_k(t) = A_k \exp\left(-\alpha \frac{4\pi^2 k^2}{L^2} t\right)$$

The initial condition is

$$\sum_{k=-\infty}^{\infty} A_k \exp\left(\frac{2\pi k}{l} ix\right)$$

Parseval's Identity

Take a function $f : \mathbb{R} \to \mathbb{C}$ with period ω. We have the fourier explansion

$$f(x) = \sum_{k=-\infty}^{\infty} a_k \exp\left(\frac{2\pi k}{\omega} ix\right).$$

Parseval's Identity:

$$\frac{1}{\omega} \int_0^{\omega} |f(x)|^2 dx = \sum_{k=-\infty}^{\infty} |a_k|^2$$

Resolving a Mystery from before

Consider the linear saw function $f(x) = x$ with period 1.

The Fourier expansion is

$$x = \sum_{k=-\infty}^{-1} -\frac{e^{2\pi kix}}{2\pi ki} + \frac{1}{2} + \sum_{k=1}^{\infty} -\frac{e^{2\pi kix}}{2\pi ki}$$

Parseval's Identity gives

$$\sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{\pi^2}{6}.$$

What about non periodic functions?

Let $f : \mathbb{R} \to \mathbb{C}$ be a function, not periodic! Can we express $f(t)$ in terms of exponential functions $e^{\zeta it}$?

Since f is not periodic, there is no condition on $\zeta \in \mathbb{R}$, so it can't just be a series.

Instead, we can look for a function $\widehat{f}(\zeta)$

$$f(t) = \int_{-\infty}^{\infty} \widehat{f}(\zeta) e^{\zeta it} d\zeta$$

What is the formula for $\widehat{f}(\zeta)$?

We can try and imitate what happened in the Fourier series case:

$$\int_{-\infty}^{\infty} f(t)e^{-\eta it}dt = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \widehat{f}(\zeta)e^{(\zeta-\eta)it}d\zeta dt$$

The right hand side doesn't converge absolutely, so we can't just swap the order of integration. If we dampen the integrand along the t-axis, then we can swap the order of integration:

$$I_\epsilon = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \widehat{f}(\zeta)e^{(\zeta-\eta)it}e^{-\epsilon^2 t^2}d\zeta dt$$

I call this the *Feynman trick*.

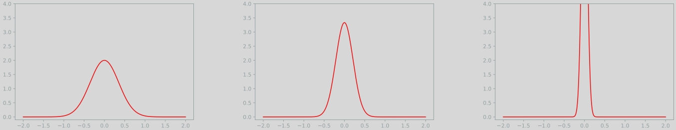
What is the formula for $\widehat{f}(\zeta)$?

$$
\begin{aligned}
I_\epsilon &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\widehat{f}(\zeta)e^{(\zeta-\eta)it}e^{-\epsilon^2 t^2}\,d\zeta dt\\
&= \int_{-\infty}^{\infty}\widehat{f}(\zeta)\int_{-\infty}^{\infty}e^{(\zeta-\eta)it}e^{-\epsilon^2 t^2}\,dtd\zeta\\
&= \sqrt{\pi}\int_{-\infty}^{\infty}\widehat{f}(\zeta)\frac{\exp\left(-\left[\frac{\zeta-\eta}{2\epsilon}\right]^2\right)}{\epsilon}\,d\zeta
\end{aligned}
$$

We can simplify further by making the following change of coordinates: $\zeta = 2x + \eta$

$$I_\epsilon = 2\sqrt{\pi}\int_{-\infty}^{\infty} \widehat{f}(2x+\eta)\frac{\exp\left(-\left[\frac{x}{\epsilon}\right]^2\right)}{\epsilon}\,dx$$

The area under the graph of $\exp\left(-\left[\frac{x}{\epsilon}\right]^2\right)/\epsilon$ is $\sqrt{\pi}$, independent of ϵ. Moreover, it gets more and more concentrated around 0 as $\epsilon \to 0$.



Therefore, $I_\epsilon \to 2\pi f(\eta)$.

The Fourier Transform

We have

$$\widehat{f}(\zeta) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(t) e^{-\zeta i t} dt$$

Notice that the Fourier transform is only going to behave well when f decays to 0 as t converges to ∞ or $-\infty$.

Fourier Transforms and Derivatives

The fourier transform interacts nicely with derivatives:

$$\widehat{\frac{df}{dt}}(\zeta) = (\zeta i)\widehat{f}(\zeta)$$

This is very useful for solving differential equations

Some interesting facts

$$f(t) = e^{-t^2} \qquad \widehat{f}(\zeta) = \frac{e^{-\zeta^2/4}}{2\sqrt{\pi}}$$

Suppose we take functions $f(t), g(t)$. We define their convolution

$$(f * g)(t) = \int_{-\infty}^{\infty} f(s)g(t-s)ds.$$

Then we have

$$\widehat{f * g}(\zeta) = 2\pi \widehat{f}(\zeta)\widehat{g}(\zeta)$$

These are the main ingredients used to prove the central limit theorem.